ECE 174

Supplemental Solutions to Homework 2

The material presented below *supplements* the solutions which you can find in the textbook's Solutions Manual (which all of you should have).

Meyer 4.1.1. Is the subset of $\mathbb{R}^n$ a vector subspace? Since $\mathbb{R}^n$ is a vector space, from the proof on page 162 of Meyer, we only have to prove closure under addition (A1) and closure under multiplication (M1). To prove these properties, we consider two arbitrary vector $\mathbf{x} = [x_1 \cdots x_n]^T$ and $\mathbf{y} = [y_1 \cdots y_n]^T$ in $\mathbb{R}^n$ and arbitrary real scalar α.

(a) $\{\mathbf{x} \,|\, x_i \geq 0\}$. This does not satisfy M1, since if we multiply by $\alpha = -1$ we go out of the set.

(b) $\{\mathbf{x} \,|\, x_1 = 0\}$. This satisfies A1 and M1 since if x_1 and y_1 are 0, then $x_1 + y_1 = 0$, and $\alpha x_1 = 0$. So it is a vector subspace.

(c) $\{\mathbf{x} \,|\, x_1 x_2 = 0\}$. This does not satisfy A1, since $(x_1 + y_1)(x_2 + y_2)$ is not necessarily 0 even if $x_1 x_2$ and $y_1 y_2$ are both 0.

(d) $\left\{\mathbf{x} \,|\, \sum_{j=1}^n x_j = 0\right\}$. This satisfies A1 and M1 since if $\sum x_i = 0$ then $\sum x_i + y_i = \sum x_i + \sum y_i = 0$, and $\sum \alpha x_i = \alpha \sum x_i = 0$. So it is a vector subspace.

(e) $\left\{\mathbf{x} \,|\, \sum_{j=1}^n x_j = 1\right\}$. This obviously does not satisfy either A1 or M1.

(f) $\{\mathbf{x} \,|\, \mathbf{A}\mathbf{x} = \mathbf{b},\, \mathbf{A}, \mathbf{b} \neq \mathbf{0}\}$. This is not a subspace. Actually, (e) is a special case of this, with $\mathbf{A} = [1\,1\, \cdots\, 1]$, $\mathbf{b} = 1$. (d) is not a special case because $\mathbf{b} = 0$. Subspaces have to contain the zero vector.

Meyer 4.1.2. Is the subset of $\mathbb{R}^{n \times n}$ a vector subspace? Addition is ordinary matrix addition, and scalar multiplication is ordinary (element-wise) multiplication by a real number. We consider arbitrary matrices $\mathbf{A}$ and $\mathbf{B}$ with elements a_{ij} and b_{ij}, and scalar α.

(a) Symmetric matrices. This satisfies A1 and M1 since if $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$, then $a_{ij} + b_{ij} = a_{ji} + b_{ji}$, and $\alpha a_{ij} = \alpha a_{ji}$.

(b) Diagonal matrices. This satisfies A1 and M1 since addition of diagonal matrices and multiplication by a constant preserve diagonality.

(c) Nonsingular matrices. This does not satisfy A1 since if $\mathbf{A}$ is nonsingular, then $-\mathbf{A}$ is nonsingular, but $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$ (the zero matrix), which is obviously singular.

(d) Singular matrices. This does not satisfy A1 since for $\mathbf{A} = [1\ 0\ ;\ 0\ 0]$ and $\mathbf{B} = [0\ 0\ ;\ 0\ 1]$ (both singular), we have $\mathbf{A} + \mathbf{B} = [1\ 0\ ;\ 0\ 1] = \mathbf{I}$, which is nonsingular.

(e) Triangular matrices. This does not satisfy A1 for $\mathbf{A}$ upper-triangular and $\mathbf{B}$ lower-triangular, $\mathbf{A} + \mathbf{B}$ is dense.

(f) Upper-triangular matrices. Similar to the case of diagonal matrices, this does satisfy A1 and M1.

(g) Matrices that commute with a given matrix $\mathbf{A}$. Yes. Take $\mathbf{B}$ and $\mathbf{C}$, satisfying $\mathbf{AB} = \mathbf{BA}$ and $\mathbf{AC} = \mathbf{CA}$. Then $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} = \mathbf{BA} + \mathbf{CA} = (\mathbf{B} + \mathbf{C})\mathbf{A}$. So A1 is satisfied. And if $\mathbf{AB} = \mathbf{BA}$, then $\mathbf{A}(\alpha\mathbf{B}) = \alpha\mathbf{AB} = \alpha\mathbf{BA} = (\alpha\mathbf{B})\mathbf{A}$, so M1 is satisfied.

(h) Matrices satisfying $\mathbf{A}^2 = \mathbf{A}$. No. If $\mathbf{A}^2 = \mathbf{A}$ and $\mathbf{B}^2 = \mathbf{B}$, then $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 = \mathbf{A} + \mathbf{B} + \mathbf{AB} + \mathbf{BA} \neq \mathbf{A} + \mathbf{B}$. So A1 is not satisfied.

(i) Matrices satisfying $\operatorname{trace}(\mathbf{A}) = 0$. Yes. This satisfies A1 and M1 since if $\operatorname{trace}(\mathbf{A}) = 0$ and $\operatorname{trace}(\mathbf{B}) = 0$, then $\operatorname{trace}(\mathbf{A} + \mathbf{B}) = \operatorname{trace}(\mathbf{A}) + \operatorname{trace}(\mathbf{B}) = 0$, and $\operatorname{trace}(\alpha\mathbf{A}) = \alpha\operatorname{trace}(\mathbf{A}) = 0$.

Meyer 4.1.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathcal{V}$.

(a) Show that $\mathcal{X} \cap \mathcal{Y}$ is a subspace. Again we only need A1 and M1. If $\mathbf{x}$ and $\mathbf{y}$ are arbitrary elements in $\mathcal{X} \cap \mathcal{Y}$, then $\mathbf{x} \in \mathcal{X}$ and $\mathbf{y} \in \mathcal{X}$, so $\mathbf{x} + \mathbf{y} \in \mathcal{X}$. Also $\mathbf{x} \in \mathcal{Y}$ and $\mathbf{y} \in \mathcal{Y}$, so $\mathbf{x} + \mathbf{y} \in \mathcal{Y}$. So $\mathbf{x} + \mathbf{y}$ is in both $\mathcal{X}$ and $\mathcal{Y}$, i.e. $\mathbf{x} + \mathbf{y} \in \mathcal{X} \cap \mathcal{Y}$. So A1 is satisfied. Also, $\alpha\mathbf{x}$ is in both $\mathcal{X}$ and $\mathcal{Y}$ if $\mathbf{x}$ is, so M1 is satisfied.

(b) Show that $\mathcal{X} \cup \mathcal{Y}$ is not necessarily a subspace. Take for example the subsets of $\mathbb{R}^2$, $\mathcal{X} = \{(x_1, x_2) \,|\, x_1 = 0\}$ and $\mathcal{Y} = \{(x_1, x_2) \,|\, x_2 = 0\}$. These are both subspaces, but $\mathcal{X} \cup \mathcal{Y}$ (all points on the coordinate axes) is not a subspace.

The proof that if $\mathcal{X}, \mathcal{Y}$ are vector subspaces of $\mathcal{V}$, then $\mathcal{X} + \mathcal{Y}$ is a vector subspace of $\mathcal{V}$ is given on pages 166-167 of Meyer.